NEW GLOBAL EXPONENTIAL STABILITY CRITERIA FOR NONLINEAR DELAY DIFFERENTIAL SYSTEMS WITH APPLICATIONS TO BAM NEURAL NETWORKS *

LEONID BEREZANSKY, †, ELENA BRAVERMAN ‡, AND LEV IDELS §

Abstract. We consider a nonlinear non-autonomous system with time-varying delays

$$\dot{x_i}(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m} F_{ij}(t, x_j(g_{ij}(t)))$$

which has a large number of applications in the theory of artificial neural networks. Via the M-matrix method, easily verifiable sufficient stability conditions for the nonlinear system and its linear version are obtained. Application of the main theorem requires just to check whether a matrix, which is explicitly constructed by the system's parameters, is an M-matrix. Comparison with the tests obtained by K. Gopalsamy (2007) and B. Liu (2013) for BAM neural networks illustrates novelty of the stability theorems. Some open problems conclude the paper.

Key words. systems of nonlinear delay differential equations, artificial neural networks, time-varying delays, global stability, M-matrix, BAM neural network, leakage delays

AMS subject classifications. 34K20, 92D25, 34K11, 34K25

1. Introduction.

One of the main motivations to study the nonlinear delay differential system

$$\dot{x_i}(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m} F_{ij}\left(t, x_j(g_{ij}(t))\right), \ t \geq 0, \ i = 1, \dots, m \tag{1.1}$$

and its linear version

$$\dot{x_i}(t) = \sum_{j=1}^{m} a_{ij}(t)x_j(g_{ij}(t)), \ i = 1, \dots, m, \tag{1.2}$$

is their importance in the study of artificial neural network models [10, 11].

For linear system (1.2) several very interesting results were obtained in [7, 9, 22, 23]. In [9] system (1.2) with constant coefficients a_{ij} was examined; in [23] the proofs were based on the assumption that a_{ij} and g_{ij} are continuous functions and $|a_{ij}(t)| \leq \beta_{ij}a_{ii}(t)$. Most of the results for system (1.1) were obtained in the case $h_i(t) \equiv t$ (see, for example, [17]). Also the requirement that all the functions involved in the system are continuous seem unduly restrictive, and we relax this assumption. In the present paper, we consider the so-called pure-delay case $h_i(t) \not\equiv t$, assuming that all parameters are measurable functions, and $F_{ij}(t, u)$ are Caratheodory functions. Via M-matrix Method we obtain novel stability results for nonlinear non-autonomous system (1.1) and linear non-autonomous system (1.2). It is to be emphasized that our technique does not require a long sequence of other theorems or conditions that must be proven or cited before the main result is justified.

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†Dept. of Math, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

‡Dept. of Math & Stats, University of Calgary, 2500 University Dr. NW, Calgary, AB, Canada T2N1N4

§Dept. of Math, Vancouver Island University (VIU), 900 Fifth St. Nanaimo, BC, Canada V9S5J5

Gopalsamy in [8] studied a model of networks known as Bidirectional Associative Memory (BAM) with leakage delays:

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -a_i x_i\left(t-\tau_i^{(1)}\right)+\sum_{j=1}^{n} a_{ij} f_j\left(y_j\left(t-\sigma_j^{(2)}\right)\right)+I_i \\ \frac{dy_i(t)}{dt} &= -b_i y_i\left(t-\tau_i^{(2)}\right)+\sum_{j=1}^{n} b_{ij} g_j\left(x_j\left(t-\sigma_j^{(1)}\right)\right)+J_i\end{aligned} \qquad i=1,\dots,n. \tag{1.3}$$

Here $\tau_i^{(k)}, \sigma_j^{(k)}$ $(k=1,2)$ are the leakage and the transmission delays accordingly. In [8] sufficient conditions for the existence of a unique equilibrium and its global stability for system (1.3) were obtained. Some interesting results for system (1.3) were obtained via the construction of Lyapunov functionals in [6, 15, 18, 19, 24, 25].

To extend and improve the results obtained in [8, 15, 16], we apply our main theorem to the non-autonomous system

$$\begin{aligned}\frac{dx_i(t)}{dt} &= r_i(t)\left[-a_i x_i\left(h_i^{(1)}(t)\right)+\sum_{j=1}^{n} a_{ij} f_j\left(y_j\left(l_j^{(2)}(t)\right)\right)+I_i\right] \\ \frac{dy_i(t)}{dt} &= p_i(t)\left[-b_i y_i\left(h_i^{(2)}(t)\right)+\sum_{j=1}^{n} b_{ij} g_j\left(x_j\left(l_j^{(1)}(t)\right)\right)+J_i\right].\end{aligned} \tag{1.4}$$

Let us quickly sketch what we accomplish here. Section 2 incorporates the main result of the paper: if a certain matrix which is explicitly constructed from the functions and the coefficients of the system is an M-matrix, then the system is globally exponentially stable. It is demonstrated that the stability condition for a nonlinear system of two equations with constant delays improves the test obtained in [8]. In Section 3 we examine stability of BAM models and obtain stability results that for a nonlinear BAM systems generalize the main theorem in [15]. Finally, Section 4 contains discussion and outlines some open problems.

2. Main Results. Consider for any $t_0 \geq 0$ the system of delay differential equations

$$\dot{x_i}(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m} F_{ij}\left(t, x_j(g_{ij}(t))\right),\ t\geq t_0,\ i=1,\dots,m, \tag{2.1}$$

with the initial conditions

$$x_i(t)=\varphi_i(t),\ t\leq t_0,\ i=1,\dots,m, \tag{2.2}$$

under the following assumptions:
(a1) a_i are Lebesgue measurable essentially bounded on $[0,\infty)$ functions, $0<\alpha_i\leq a_i(t)\leq A_i$;
(a2) $F_{ij}(t,\cdot)$ are continuous functions, $F_{ij}(\cdot,u)$ are measurable locally essentially bounded functions, $|F_{ij}(t,u)|\leq L_{ij}|u|, t\geq 0$;
(a3) h_i, g_{ij} are measurable functions, $0\leq t-h_i(t)\leq \tau_i,\ 0\leq t-g_{ij}(t)\leq \sigma$;
(a4) φ_i are continuous functions on $[t_0-\sigma, t_0]$, where
$\sigma=\max\{\tau_k, \sigma_{ij}, k, i, j=1,\dots,m\}$.

Henceforth assume that conditions $(a1)-(a4)$ hold for problem (2.1), (2.2) and its modifications, and the problem has a unique solution.

We will use some traditional notations. A matrix $B = (b_{ij})_{i,j=1}^m$ is nonnegative if $b_{ij} \geq 0$ and positive if $b_{ij} > 0$, $i,j = 1,\dots,m$; $\|a\|$ is a norm of a column vector $a = (a_1,\dots,a_m)^T$ in $\mathbb{R}^m$; $\|B\|$ is the corresponding matrix norm of a matrix B, $|a| = (|a_1|,\dots,|a_m|)^T$ and $|B| = (|b_{ij}|)_{i,j=1}^m$. As usual, function $X(t) = (x_1(t),\dots,x_m(t))^T$ is a solution of (2.1), (2.2) if it satisfies (2.1) almost everywhere for $t > t_0$ and (2.2) for $t \leq t_0$. Problem (2.1),(2.2) has a unique global solution on $[t_0,\infty)$, if, for example, we assume along with $(a1)-(a4)$ that functions $F_{ij}(t,u)$ are locally Lipschitz in u. The following classical definition of an $M-$matrix will be used.

Definition 2.1. *[5] A matrix $B = (b_{ij})_{i,j=1}^m$ is called a (nonsingular) M-*matrix *if $b_{ij} \leq 0, i \neq j$ and one of the following equivalent conditions holds:*
- there exists a positive inverse matrix $B^{-1} > 0$;
- the principal minors of matrix B are positive.

Lemma 2.2. *[5] B is an $M-$matrix if $b_{ij} \leq 0, i \neq j$ and at least one of the following conditions holds:*
- $b_{ii} > \sum_{j\neq i} |b_{ij}|,\ i = 1,\dots,m$;
- $b_{jj} > \sum_{i\neq j} |b_{ij}|,\ j = 1,\dots,m$;
- there exist positive numbers $\xi_i, i = 1,\dots,m$ such that $\xi_i b_{ii} > \sum_{j\neq i} \xi_j |b_{ij}|,\ i = 1,\dots,m$;
- there exist positive numbers $\xi_i, i = 1,\dots,m$ such that $\xi_j b_{jj} > \sum_{i\neq j} \xi_i |b_{ij}|,\ j = 1,\dots,m$.

Definition 2.3. *System (2.1) is globally exponentially stable if there exist $M > 0, \lambda > 0$ such that for any solution $X(t)$ of problem (2.1),(2.2) the inequality*

$$\|X(t)\| \leq M e^{-\lambda(t-t_0)} \left(\|x(t_0)\| + \sup_{t<t_0} \|\varphi(t)\| \right)$$

holds, where M and λ do not depend on t_0. We define matrix C as follows

$$C = (c_{ij})_{i,j=1}^m, \quad c_{ii} = 1 - \frac{A_i(A_i + L_{ii})\tau_i + L_{ii}}{\alpha_i},\ c_{ij} = -\frac{A_i L_{ij}\tau_i + L_{ij}}{\alpha_i}, i \neq j. \quad (2.3)$$

Theorem 2.4. *Suppose C defined by (2.3) is an M-matrix. Then system (2.1) is globally exponentially stable.*

Proof. The solution $X(t) = (x_1(t),\dots,x_n(t))^T$ of problem (2.1),(2.2) is also a solution of the problem

$$\dot{x_i}(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^m F_{ij}\left(t, x_j(g_{ij}(t)) + \varphi_j(g_{ij}(t))\right) - a_i(t)\varphi_i(h_i(t)),\ t \geq t_0, \quad (2.4)$$

$i = 1,\dots,m$, where we assume that $x_i(t) = 0$, $t < t_0$ and $\varphi_i(t) = 0$ for $t \geq t_0$. After the substitution $x_i(t) = e^{-\lambda(t-t_0)} y_i(t),\ t \geq t_0$, where $0 < \lambda < \min_i \alpha_i$, equation (2.4) has the form

$$\begin{aligned} \dot{y_i}(t) = \quad & \lambda y_i(t) - e^{\lambda(t-h_i(t))} a_i(t) y_i(h_i(t)) \\ & + \sum_{j=1}^m e^{\lambda(t-t_0)} F_{ij}\left(t, e^{-\lambda(g_{ij}(t)-t_0)} y_j(g_{ij}(t)) + \varphi_j(g_{ij}(t))\right) \qquad (2.5) \\ & - e^{\lambda(t-t_0)} a_i(t)\varphi_i(h_i(t)). \end{aligned}$$

After denoting $\mu_i(t) := e^{\lambda(t-h_i(t))}a_i(t) - \lambda$, equation (2.5) can be rewritten as

$$\begin{aligned}\dot{y_i}(t) = &-\mu_i(t)y_i(t) + e^{\lambda(t-h_i(t))}a_i(t)\int_{h_i(t)}^{t}\dot{y_i}(s)ds\\ &+\sum_{j=1}^{m} e^{\lambda(t-t_0)}F_{ij}\left(t, e^{-\lambda(g_{ij}(t)-t_0)}y_j(g_{ij}(t)) + \varphi_j(g_{ij}(t))\right)\\ &-e^{\lambda(t-t_0)}a_i(t)\varphi_i(h_i(t)).\end{aligned}$$

For $\dot{y_i}(s)$ we substitute the right-hand side of equation (2.5)

$$\begin{aligned}\dot{y_i}(t) = &-\mu_i(t)y_i(t) + e^{\lambda(t-h_i(t))}a_i(t)\int_{h_i(t)}^{t}\Bigg[\lambda y_i(s) - e^{\lambda(s-h_i(s))}a_i(s)y_i(h_i(s))\\ &+\sum_{j=1}^{m} e^{\lambda(s-t_0)}F_{ij}\left(s, e^{-\lambda(g_{ij}(s)-t_0)}y_j(g_{ij}(s)) + \varphi_j(g_{ij}(s))\right) - e^{\lambda(s-t_0)}a_i(t)\varphi_i(h_i(s))\Bigg]ds\\ &+\sum_{j=1}^{m} e^{\lambda(t-t_0)}F_{ij}\left(t, e^{-\lambda g_{ij}(t)-t_0)}y_j(g_{ij}(t)) + \varphi_j(g_{ij}(t))\right) - e^{\lambda(t-t_0)}a_i(t)\varphi_i(h_i(t)).\end{aligned}$$

Hence

$$\begin{aligned}y_i(t) = &\ e^{-\int_{t_0}^{t}\mu_i(s)ds}x_i(t_0) + \int_{t_0}^{t} e^{-\int_s^t \mu_i(\zeta)\,d\zeta}\Bigg(e^{\lambda(s-h_i(s))}a_i(t)\int_{h_i(s)}^{s}\Bigg[\lambda y_i(\zeta) - e^{\lambda(\zeta-h_i(\zeta))}a_i(\zeta)y_i(h_i(\zeta))\\ &+\sum_{j=1}^{m} e^{\lambda(\zeta-t_0)}F_{ij}\left(\zeta, e^{-\lambda(g_{ij}(\zeta)-t_0)}y_j(g_{ij}(\zeta)) + \varphi_j(g_{ij}(\zeta))\right) - e^{\lambda(\zeta-t_0)}a_i(\zeta)\varphi_i(h_i(\zeta))\Bigg]d\zeta\\ &+\sum_{j=1}^{m} e^{\lambda(s-t_0)}F_{ij}\left(s, e^{-\lambda(g_{ij}(s)-t_0)}y_j(g_{ij}(s)) + \varphi_j(g_{ij}(s))\right) - e^{\lambda(s-t_0)}a_i(s)\varphi_i(h_i(s))\Bigg)ds.\end{aligned}$$

Then

$$\begin{aligned}|y_i(t)| \le &\ |x_i(t_0)| + \int_{t_0}^{t} e^{-\int_s^t \mu_i(\zeta)d\zeta}\mu_i(s)\Bigg(A_ie^{\lambda\tau_i}\int_{h_i(s)}^{s}\Bigg[\lambda|y_i(\zeta)|\\ &+e^{\lambda\tau_i}A_i|y_i(h_i(\zeta))| + \sum_{j=1}^{m} e^{\lambda\sigma}L_{ij}|y_j(g_{ij}(\zeta))| + \left(\sum_{j=1}^{m}L_{ij}e^{\lambda\sigma} + e^{\lambda\tau_i}A_i\right)\|\varphi_i\|\Bigg]d\zeta\\ &+\sum_{j=1}^{m} e^{\lambda\sigma}L_{ij}|y_j(g_{ij}(s))| + \left(\sum_{j=1}^{m}L_{ij} + e^{\lambda\tau_i}A_i\right)\|\varphi_i\|\Bigg)\Bigg/\mu_i(s)\,ds,\end{aligned}$$

where $\|\varphi_i\| = \sup_{t_0-\sigma\le t\le t_0}|\varphi_i(t)|$. Let $\overline{y}_i = \max_{t_0\le t\le b}|y_i(t)|$. If we fix some $b > t_0$ and denote $\overline{Y} = (\overline{y}_1, \ldots, \overline{y}_n)^T$ then

$$\begin{aligned}\overline{y}_i \le &\ |x_i(t_0)| + \frac{(\sum_{j=1}^{m}L_{ij}e^{\lambda\sigma} + e^{\lambda\tau_i}A_i)(A_ie^{\lambda\tau_i}\tau_i + 1)}{\alpha_i - \lambda}\|\varphi_i\|\\ &+\left[A_i\tau_ie^{\lambda\tau_i}\left(\lambda\overline{y}_i + A_ie^{\lambda\tau_i}\overline{y}_i + \sum_{j=1}^{m}e^{\lambda\sigma}L_{ij}\overline{y}_j\right) + \sum_{j=1}^{m}e^{\lambda\sigma}L_{ij}\overline{y}_j\right]\Bigg/(\alpha_i - \lambda).\end{aligned}$$

We define the matrix $C(\lambda) = (c_{ij}(\lambda))_{i,j=1}^m$ with the entries

$$c_{ii}(\lambda) = 1 - \frac{A_i e^{\lambda \tau_i}(\lambda + A_i e^{\lambda \tau_i} + e^{\lambda \sigma_{ii}} L_{ii})\tau_i + e^{\lambda \sigma_{ii}} L_{ii}}{\alpha_i - \lambda} \ ,$$

$$c_{ij}(\lambda) = -\frac{A_i e^{\lambda \tau_i} e^{\lambda \sigma} L_{ij} \tau_i + e^{\lambda \sigma} L_{ij}}{\alpha_i - \lambda}, \ i \neq j.$$

Clearly, the vector inequality $C(\lambda)\overline{Y} \leq |X(t_0)| + M_1(\lambda)|\varphi|$ is valid for $t_0 \leq t \leq b$, where

$$M_1(\lambda) = \max_{1 \leq i \leq m} \frac{(\sum_{j=1}^m L_{ij} e^{\lambda \sigma} + e^{\lambda \tau_i A_i})(A_i e^{\lambda \tau_i} \tau_i + 1)}{\alpha_i - \lambda},$$

and we have $\lim_{\lambda \to 0} C(\lambda) = C(0) = C$. By the assumption of the theorem, $C(0)$ is an M-matrix. For $0 < \lambda < \min_i \alpha_i$ the entries of the matrix $C(\lambda)$ are continuous functions; therefore, the determinant of this matrix is a continuous function. For some small $\lambda > 0$ all the principal minors of $C(\lambda)$ are positive; the latter along with $c_{ij}(\lambda) \leq 0$, $i \neq j$ implies that $C(\lambda)$ is an M-matrix for small λ. If we fix such parameter $\lambda = \lambda_0$, then for $\overline{Y}$ there is an *a priori* estimate

$$\|\overline{Y}\| \leq M(\|X(t_0)\| + \|\varphi\|), \quad M = \|C^{-1}(\lambda_0)\| \ \max\{1, M_1(\lambda_0)\},$$

where M does not depend on b and t_0. Finally, $X(t) = e^{-\lambda_0 (t - t_0)} Y(t)$, hence

$$\|X(t)\| \leq M\left(\|X(t_0)\| + \|\varphi\|\right) e^{-\lambda_0 (t - t_0)},$$

which completes the proof. □

Consider the system with off-diagonal nonlinearities

$$\dot{x_i}(t) = -a_i(t) x_i(h_i(t)) + \sum_{j \neq i} F_{ij}\left(t, x_j(g_{ij}(t))\right), \ t \geq 0, \ i = 1, \dots, m. \tag{2.6}$$

Corollary 1. *Suppose that the matrix C defined by*

$$C = (c_{ij})_{i,j=1}^m, \quad c_{ii} = 1 - \frac{A_i^2 \tau_i}{\alpha_i}, \ c_{ij} = -\frac{A_i L_{ij} \tau_i + L_{ij}}{\alpha_i}, i \neq j, \tag{2.7}$$

is an M-matrix. Then system (2.6) is globally exponentially stable.

The next corollary examines the system with a non-delay linear term

$$\dot{x_i}(t) = -a_i(t) x_i(t) + \sum_{j=1}^m F_{ij}\left(t, x_j(g_{ij}(t))\right), \ t \geq 0, \ i = 1, \dots, m. \tag{2.8}$$

$$B = (b_{ij})_{i,j=1}^m, \quad b_{ii} = 1 - \frac{L_{ii}}{\alpha_i}, \quad b_{ij} = -\frac{L_{ij}}{\alpha_i}, \ i \neq j. \tag{2.9}$$

Corollary 2. *Suppose B defined by (2.9) is an M-matrix. Then system (2.8) is globally exponentially stable.*

For the delay linear system

$$\dot{x_i}(t) = \sum_{j=1}^{m} a_{ij}(t)x_j(g_{ij}(t)),\ i = 1,\dots,m, \tag{2.10}$$

assume that a_{ij} are essentially bounded on $[0,\infty)$ functions, $0 < \alpha_i \le -a_{ii}(t) \le A_i, |a_{ij}(t)| \le A_{ij}, i \ne j$, g_{ij} are measurable functions, $0 \le t - g_{ij}(t) \le \sigma_{ij}$. Denote

$$D = (d_{ij})_{i,j=1}^m,\quad d_{ii} = 1 - \frac{A_i^2\sigma_{ii}}{\alpha_i},\ d_{ij} = -\frac{A_iA_{ij}\sigma_{ii} + A_{ij}}{\alpha_i},\ i \ne j. \tag{2.11}$$

Corollary 3. *Suppose D defined by (2.11) is an M-matrix. Then system (2.10) is exponentially stable.*

The same result holds for the linear system with non-delay diagonal terms

$$\dot{x_i}(t) = -a_i(t)x_i(t) + \sum_{j\ne i} a_{ij}(t)x_j(g_{ij}(t)),\ i = 1,\dots,m, \tag{2.12}$$

where a_{ij} are essentially bounded on $[0,\infty)$ functions, $0 < \alpha_i \le a_i(t) \le A_i, |a_{ij}(t)| \le A_{ij}, i \ne j$, g_{ij} are measurable functions, $0 \le t - g_{ij}(t) \le \sigma$. Denote

$$F = (f_{ij})_{i,j=1}^m,\quad f_{ii} = 1,\ f_{ij} = -\frac{A_{ij}}{\alpha_i},\ i \ne j. \tag{2.13}$$

Corollary 4. *Suppose F defined by (2.13) is an M-matrix. Then system (2.12) is exponentially stable.*

Corollary 5. *Suppose $m = 2$, $A_1(A_1 + L_{11})\tau_1 + L_{11} < \alpha_1$ and*

$$(\alpha_1 - A_1(A_1+L_{11})\tau_1 - L_{11})(\alpha_2 - A_2(A_2+L_{22})\tau_2 - L_{22}) > L_{12}L_{21}(1+A_1\tau_1)(1+A_2\tau_2). \tag{2.14}$$

Then system (2.1) is globally exponentially stable.

Proof. For $m = 2$ the matrix C denoted by (2.3) has the form

$$C = \begin{pmatrix} 1 - \frac{A_1(A_1+L_{11})\tau_1+L_{11}}{\alpha_1} & -\frac{A_1L_{12}\tau_1+L_{12}}{\alpha_1} \\ -\frac{A_2L_{21}\tau_2+L_{21}}{\alpha_2} & 1 - \frac{A_2(A_2+L_{22})\tau_2+L_{22}}{\alpha_2} \end{pmatrix}.$$

The off-diagonal entries are negative, by the assumption of the corollary the principal minors are positive, so C is an M-matrix. □

Corollary 6. *Suppose $m = 2, L_{11} < \alpha_1$ and $(\alpha_1 - L_{11})(\alpha_2 - L_{22}) > L_{12}L_{21}$. Then system (2.8) is globally exponentially stable.*

Corollary 7. *Suppose $m = 2$, $\sigma_{11} < \alpha_1/A_1^2$ and the inequality*

$$(\alpha_1 - A_1^2\sigma_{11})(\alpha_2 - A_2^2\sigma_{22}) > A_{12}A_{21}(1 + A_1\sigma_{11})(1 + A_2\sigma_{22})$$

holds. Then system (2.10) is exponentially stable.

Corollary 8. *Suppose $m = 2$ and $\alpha_1\alpha_2 > A_{12}A_{21}$. Then system (2.12) is exponentially stable.*

Remark 1. *By Corollary 8, the linear system*

$$\begin{aligned} \dot{x}(t) &= -a_{11}x(t) + a_{12}y(t) \\ \dot{y}(t) &= a_{21}x(t) - a_{22}y(t) \end{aligned} \tag{2.15}$$

with the coefficients $a_{ii} > 0$ is exponentially stable if

$$a_{11}a_{22} > a_{12}a_{21}. \tag{2.16}$$

Condition (2.16) is necessary and sufficient for exponential stability of system (2.15); therefore, Theorem 2.4 and its corollaries cannot be improved.

In the paper [8] Gopalsamy considered autonomous system (1.3). For $n = 1$ it has the form

$$\begin{aligned} \dot{x}(t) &= -a_1 x(t-\tau_1) + a_{12} f_1(y(t-\sigma_1)) \\ \dot{y}(t) &= -a_2 y(t-\tau_2) + a_{21} f_2(x(t-\sigma_2)) \end{aligned} \tag{2.17}$$

where $a_i > 0, a_{ij} > 0, \tau_i \geq 0, \sigma_i \geq 0, |f_i(u)| \leq L_i|u|$ *and* $i = 1, 2$. In [8] the following global attractivity result was obtained. If $a_i\tau_i < 1$ and

$$\frac{1-a_1\tau_1}{1+a_1\tau_1} > \frac{a_{12}L_1}{a_1}, \ \frac{1-a_2\tau_2}{1+a_2\tau_2} > \frac{a_{21}L_2}{a_2} \tag{2.18}$$

then any solution of system (2.17) tends to zero. By Corollary 5 equation (2.17) is exponentially stable if $a_i\tau_i < 1$ and

$$\frac{(1-a_1\tau_1)(1-a_2\tau_2)}{(1+a_1\tau_1)(1+a_2\tau_2)} > \frac{a_{12}a_{21}L_1L_2}{a_1a_2}. \tag{2.19}$$

Obviously condition (2.18) implies (2.19).

Example 1. *Consider system (2.17) where $a_1 = 0.8$, $a_2 = 0.5$, $a_{12} = a_{21} = 1$, $\tau_1 = 0.5$, $\tau_2 = 0.4$, $|f_i(u)| \leq L_i|u|$ with $L_1 = 0.5$, $L_2 = 0.2$, $\sigma_i \geq 0$. Here the first inequality in (2.18) does not hold since*

$$\frac{1-a_1\tau_1}{1+a_1\tau_1} = \frac{3}{7} < \frac{5}{8} = \frac{a_{12}L_1}{a_1},$$

and therefore the result of [8] cannot be applied. However, $a_1\tau_1 = 0.4 < 1$ and inequality (2.19)

$$\frac{(1-a_1\tau_1)(1-a_2\tau_2)}{(1+a_1\tau_1)(1+a_2\tau_2)} = \frac{2}{7} > \frac{1}{4} = \frac{a_{12}a_{21}L_1L_2}{a_1a_2}$$

holds, thus Corollary 5 implies exponential stability, hence for $n = 1$ ($m = 2$) we obtained the result which is sharper than the relevant result in [8].

In the next section, we provide more in-depth analysis of systems with leakage delays which include (2.17) as a special case.

3. BAM Network with Time-Varying Delays. In [8] a class (1.3) of BAM neural networks with leakage (forgetting) delays was under study. Via Lyapunov functionals method sufficient conditions for the existence of a unique equilibrium and its global stability for system (1.3) were obtained. To extend and improve the results obtained in [8] and [15, 16], we will focus on the non-autonomous BAM neural network model

$$\begin{aligned} \frac{dx_i(t)}{dt} &= r_i(t)\left(-a_i x_i(h_i^{(1)}(t)) + \sum_{j=1}^{n} a_{ij} f_j\left(y_j\left(l_j^{(2)}(t)\right)\right) + I_i\right) \\ \frac{dy_i(t)}{dt} &= p_i(t)\left(-b_i y_i(h_i^{(2)}(t)) + \sum_{j=1}^{n} b_{ij} g_j\left(x_j\left(l_j^{(1)}(t)\right)\right) + J_i\right), \end{aligned} \tag{3.1}$$

$i = 1, \dots, n$, $t \geq 0$, with the initial conditions

$$x_i(t) = \varphi_i(t), \quad y_i(t) = \varphi_{i+n}(t), \quad t < 0, \quad i = 1, \dots, n. \tag{3.2}$$

The following auxiliary lemma will be used.

LEMMA 3.1. *Let*

$$u_i = \sum_{j=1}^{m} F_{ij}(u_j), \ i = 1, \dots, m, \tag{3.3}$$

where $|F_{ij}(u) - F_{ij}(v)| \leq L_{ij}|u - v|$*, the matrix* L *be* $L = (L_{ij})_{i,j=1}^{m}$ *and denote by* $r(L)$ *the spectral radius of* L*. If* $r(L) < 1$ *then system (3.3) has a unique solution.*

Proof. Consider the operator $T : \mathbb{R}^m \to \mathbb{R}^m$ denoted by

$$T(u) := T((u_1, \dots, u_m)^T) = \left(\sum_{j=1}^{m} F_{1j}(u_j), \dots, \sum_{j=1}^{m} F_{mj}(u_j) \right)^T .$$

Then

$$|T(u) - T(v)| \leq \left(\sum_{j=1}^{m} L_{1j}|u_j - v_j|, \dots, \sum_{j=1}^{m} L_{mj}|u_j - v_j| \right)^T = L|u - v|.$$

It is well known that $r(L) = \inf_{\|\cdot\|} \|L\|$, where the infimum is taken on all (equivalent) norms in $\mathbb{R}^m$. Since $r(L) < 1$, we can choose a norm in $\mathbb{R}^m$ such that the corresponding norm $\|L\| \leq q < 1$. We fix now such a norm and have

$$\|T(u) - T(v)\| \leq \|L\| \|u - v\| \leq q < 1.$$

By the Banach contraction principle the equation $u = T(u)$ has a unique solution. □

COROLLARY 9. *Suppose at least one of the following conditions holds:*
1. $\max |\lambda(L)| < 1$*, where the maximum is taken over all eigenvalues of matrix* L*.*
2. $\max_i \sum_{j=1}^{m} L_{ij} < 1$.
3. $\max_j \sum_{i=1}^{m} L_{ij} < 1$.
4. $\sum_{i=1}^{m} \sum_{j=1}^{m} L_{ij}^2 < 1$.
Then system (3.3) has a unique solution.

It should be noted that the proof of Lemma 3.1 is original and shorter than, for example, the recently published proof [21, Theorem 2.2].

Henceforth, assume that the following assumptions hold for (3.1), (3.2):
(b1) r_i, p_i are Lebesgue measurable essentially bounded on $[0, \infty)$ functions, $0 < \alpha_i \leq r_i(t) \leq R_i, 0 < \beta_i \leq p_i(t) \leq P_i$;
(b2) $f_j(\cdot), g_j(\cdot)$ are continuous functions; $|f_j(u) - f_j(v)| \leq L_j^f |u - v|, |g_j(u) - g_j(v)| \leq L_j^g |u - v|$;
(b3) $h_i^{(1)}, h_i^{(2)}, l_j^{(1)}, l_j^{(2)}$ are Lebesgue measurable functions, $0 \leq t - h_i^{(1)}(t) \leq \tau_i^{(1)}, \ 0 \leq t - h_i^{(2)}(t) \leq \tau_i^{(2)}$,
$0 \leq t - l_i^{(1)}(t) \leq \sigma_i^{(1)}, \ 0 \leq t - l_i^{(2)}(t) \leq \sigma_i^{(2)}$;
(b4) φ_i are continuous functions.

Let $(x^*, y^*) = (x_1^*, \dots, x_n^*, y_1^*, \dots, y_n^*)$ be a solution of the system

$$\begin{aligned} a_i x_i &= \sum_{j=1}^{n} a_{ij} f_j(y_j) + I_i \\ b_i y_i &= \sum_{j=1}^{n} b_{ij} g_j(x_j) + J_i. \end{aligned} \tag{3.4}$$

Apparently the existence of a solution of system (3.4) is equivalent to the existence of the solution of the following system

$$\begin{aligned} u_i &= \sum_{j=1}^{n} a_{ij} f_j\left(\frac{v_j}{b_j}\right) + I_i \\ v_i &= \sum_{j=1}^{n} b_{ij} g_j\left(\frac{u_j}{a_j}\right) + J_i. \end{aligned} \tag{3.5}$$

Denoting $u_j = x_j,\ j = 1, \dots, n; u_j = y_{j-n},\ j = n+1, \dots, 2n$,

$$F_{ij}(u) = \begin{cases} \frac{a_{i,j-n}}{a_i} f_{j-n}(u) + \frac{I_i}{a_i}, & i = 1, \dots, n; j = n+1, \dots, 2n, \\ \frac{b_{i-n,j}}{b_{i-n}} g_j(u) + \frac{J_{i-n}}{a_{i-n}}, & i = n+1, \dots, 2n; j = 1, \dots, n, \\ 0, & \textit{otherwise}, \end{cases}$$

we can rewrite system (3.4) in the form of (3.3) with $m = 2n$, $|F_{ij}(u) - F_{ij}(v)| \le L_{ij}|u - v|$.

We introduce the matrix $A = (L_{ij})_{i,j=1}^{2n}$

$$A = \begin{pmatrix} 0 & \dots & 0 & \frac{|a_{11}|L_1^f}{a_1} & \dots & \frac{|a_{1n}|L_n^f}{a_1} \\ - & - & - & - & - & - \\ 0 & \dots & 0 & \frac{|a_{n1}|L_1^f}{a_n} & \dots & \frac{|a_{nn}|L_n^f}{a_n} \\ \frac{|b_{11}|L_1^g}{b_1} & \dots & \frac{|b_{1n}|L_n^g}{b_1} & 0 & \dots & 0 \\ - & - & - & - & - & - \\ \frac{|b_{n1}|L_1^g}{b_n} & \dots & \frac{|b_{nn}|L_n^g}{b_n} & 0 & \dots & 0 \end{pmatrix}.$$

By the same token we can rewrite (3.5) in the form (3.3), where

$$F_{ij}(u) = \begin{cases} a_{i,j-n} f_{j-n}(\frac{u}{b_{j-n}}) + I_i, & i = 1, \dots, n; j = n+1, \dots, 2n, \\ b_{i-n,j} g_j(\frac{u}{a_j}) + J_{i-n}, & i = n+1, \dots, 2n; j = 1, \dots, n, \\ 0, & \textit{otherwise}, \end{cases}$$

with $m = 2n$, $|F_{ij}(u) - F_{ij}(v)| \le L_{ij}|u - v|$, and introduce the matrix $B = (L_{ij})_{i,j=1}^{2n}$

$$B = \begin{pmatrix} 0 & \dots & 0 & \frac{|a_{11}|L_1^f}{b_1} & \dots & \frac{|a_{1n}|L_n^f}{b_n} \\ - & - & - & - & - & - \\ 0 & \dots & 0 & \frac{|a_{n1}|L_1^f}{b_1} & \dots & \frac{|a_{nn}|L_n^f}{b_n} \\ \frac{|b_{11}|L_1^g}{a_1} & \dots & \frac{|b_{1n}|L_n^g}{a_n} & 0 & \dots & 0 \\ - & - & - & - & - & - \\ \frac{|b_{n1}|L_1^g}{a_1} & \dots & \frac{|b_{nn}|L_n^g}{a_n} & 0 & \dots & 0 \end{pmatrix}.$$

In the following theorem we apply Lemma 3.1 to systems (3.4) and (3.5) with $L = A$ and $L = B$, and obtain conditions $1-4$ and $5-8$, respectively.

THEOREM 3.2. *Suppose at least one of the following conditions holds:*

1. $\max|\lambda(A)| < 1$, *where maximum is taken on all eigenvalues of matrix* A.

2. $\max_i \sum_{j=1}^n \frac{|a_{ij}|L_j^f}{a_i} < 1, \quad \max_i \sum_{j=1}^n \frac{|b_{ij}|L_j^g}{b_i} < 1.$

3. $\max_j \sum_{i=1}^n \frac{|a_{ij}|L_j^f}{a_i} < 1, \quad \max_j \sum_{i=1}^n \frac{|b_{ij}|L_j^g}{b_i} < 1.$

4. $\sum_{i=1}^n \sum_{j=1}^n \left[\left(\frac{|a_{ij}|L_j^f}{a_i} \right)^2 + \left(\frac{|b_{ij}|L_j^g}{b_i} \right)^2 \right] < 1.$

5. $\max|\lambda(B)| < 1$, *where maximum is taken on all eigenvalues of matrix* B.

6. $\max_i \sum_{j=1}^n \frac{|a_{ij}|L_j^f}{b_j} < 1, \quad \max_i \sum_{j=1}^n \frac{|b_{ij}|L_j^g}{a_j} < 1.$

7. $\max_j \sum_{i=1}^n \frac{|a_{ij}|L_j^f}{b_j} < 1, \quad \max_j \sum_{i=1}^n \frac{|b_{ij}|L_j^g}{a_j} < 1.$

8. $\sum_{i=1}^n \sum_{j=1}^n \left[\left(\frac{|a_{ij}|L_j^f}{b_j} \right)^2 + \left(\frac{|b_{ij}|L_j^g}{a_j} \right)^2 \right] < 1.$

Then system (3.4) has a unique solution and thus system (3.1) has a unique equilibrium.

REMARK 2. *Note that the conclusion of Theorem 3.2 under condition 7 was obtained in paper [8].*

Below, assume that system (3.1) has a unique equilibrium (x^*, y^*). To obtain a global stability condition for this equilibrium, consider the matrix $C_{BAM} = (c_{ij})_{i,j=1}^{2n}$, where

$$c_{ii} = \begin{cases} 1 - a_i R_i^2 \tau_i^{(1)}/\alpha_i, & i = 1, \dots, n, \\ 1 - b_{i-n} P_{i-n}^2 \tau_{i-n}^{(2)}/\beta_{i-n}, & i = n+1, \dots, 2n, \end{cases} \tag{3.6}$$

$$c_{ij} = \begin{cases} -|a_{i,j-n}| R_i L_{j-n}^f (a_i R_i \tau_i^{(1)} + 1)/(\alpha_i a_i), \ i = 1, \dots, n, j = n+1, \dots, 2n, \\ -|b_{i-n,j}| P_{i-n} L_j^g (b_{i-n} P_{i-n} \tau_{i-n}^{(2)} + 1)/(\beta_{i-n} b_{i-n}), \ i = n+1, \dots, 2n, j = 1, \dots, n, \\ 0, \ \textit{otherwise}. \end{cases} \tag{3.7}$$

THEOREM 3.3. *Suppose matrix* C_{BAM} *is an M-matrix. Then the equilibrium* (x^*, y^*) *of system (3.1) is globally exponentially stable.*

Proof. After the substitution $x_i(t) = u_i(t) + x_i^*, y_i(t) = v_i(t) + y_i^*$, system (3.1) has the form

$$\begin{aligned} \dot{u}_i(t) &= -r_i(t) a_i u_i(h_i^{(1)}(t)) + \sum_{j=1}^n a_{ij} r_i(t) \left(f_j(v_j(l_j^{(2)}(t)) + y_j^*) - f_j(y_j^*) \right) \\ \dot{v}_i(t) &= -p_i(t) b_i v_i(h_i^{(2)}(t)) + \sum_{j=1}^n b_{ij} p_i(t) \left(g_j(u_j(l_j^{(1)}(t)) + x_j^*) - g_j(x_j^*) \right), \end{aligned} \tag{3.8}$$

$$x_i(t) = \begin{cases} u_i(t), & i = 1, \ldots, n, \\ v_{i-n}(t), & i = n+1, \ldots, 2n, \end{cases} \qquad a_i(t) = \begin{cases} r_i(t) a_i, & i = 1, \ldots, n, \\ p_{i-n}(t) b_{i-n}, & i = n+1, \ldots, 2n, \end{cases}$$

$$h_i(t) = \begin{cases} h_i^{(1)}(t), & i = 1, \ldots, n, \\ h_{i-n}^{(2)}(t), & i = n+1, \ldots, 2n, \end{cases}$$

$$g_{i,j}(t) = \begin{cases} l_{j-n}^{(2)}(t), & i = 1, \ldots, n, j = n+1, \ldots, 2n, \\ l_j^{(1)}(t), & i = n+1, \ldots, 2n, j = 1, \ldots, n, \\ 0, & \textit{otherwise}, \end{cases}$$

$$F_{i,j}(t,x) = \begin{cases} a_{i,j-n} r_i(t) \left(f_{j-n}(x + y_{j-n}^*) - f_{j-n}(y_{j-n}^*) \right), \ i = 1, \ldots, n, j = n+1, \ldots, 2n, \\ b_{i-n,j} p_{i-n}(t) \left(g_j(x + x_j^*) - g_j(x_j^*) \right), \ i = n+1, \ldots, 2n, j = 1, \ldots, n, \\ 0, \ \textit{otherwise}. \end{cases}$$

We have $0 < \alpha_i \leq a_i(t) \leq A_i$, where

$$\alpha_i = \begin{cases} r_i a_i, & i = 1, \ldots, n, \\ p_{i-n} b_{i-n}, & i = n+1, \ldots, 2n, \end{cases} \qquad A_i = \begin{cases} R_i a_i, & i = 1, \ldots, n, \\ P_{i-n} b_{i-n}, & i = n+1, \ldots, 2n, \end{cases}$$

and $|F_{ij}(t,x)| \leq L_{ij}|x|$ with the constant

$$L_{i,j} = \begin{cases} |a_{i,j-n}| R_i L_{j-n}^f, & i = 1, \ldots, n, j = n+1, \ldots, 2n, \\ |b_{i-n,j}| P_{i-n} L_j^g, & i = n+1, \ldots, 2n, j = 1, \ldots, n, \\ 0, & \textit{otherwise}. \end{cases}$$

System (3.8) with $m = 2n$ has form (2.6), where matrix C_{BAM} corresponds to matrix C defined by (2.7). All conditions of Corollary 1 hold; therefore, the trivial solution of system (3.8) is globally exponentially stable; hence the equilibrium (x^*, y^*) of system (3.1) is globally exponentially stable. ∎

Corollary 10. *Suppose at least one of the following conditions holds:*

1. $\displaystyle\sum_{j=1}^n \frac{|a_{ij}| R_i L_j^f (a_i R_i \tau_i^{(1)} + 1)}{\alpha_i a_i} < 1 - \frac{a_i R_i^2 \tau_i^{(1)}}{\alpha_i},$

$\displaystyle\sum_{j=1}^n \frac{|b_{ij}| P_i L_j^g (b_i P_i \tau_i^{(2)} + 1)}{\beta_i b_i} < 1 - \frac{b_i P_i^2 \tau_i^{(2)}}{\beta_i}, \quad i = 1, \ldots, n.$

2. $\displaystyle\sum_{i=1}^n \frac{|a_{ij}| R_i L_j^f (a_i R_i \tau_i^{(1)} + 1)}{\alpha_i a_i} < 1 - \frac{b_j P_j^2 \tau_j^{(2)}}{\beta_j},$

$\displaystyle\sum_{i=1}^n \frac{|b_{ij}| P_i L_j^g (b_i P_i \tau_i^{(2)} + 1)}{\beta_i b_i} < 1 - \frac{a_j R_j^2 \tau_j^{(1)}}{\alpha_j}, \quad j = 1, \ldots, n.$

3. There exist positive numbers $\mu_k, k = 1, \ldots, 2n$ such that

$$\sum_{j=1}^n \frac{\mu_{j+n} |a_{ij}| R_i L_j^f (a_i R_i \tau_i^{(1)} + 1)}{\alpha_i a_i} < \mu_i \left(1 - \frac{a_i R_i^2 \tau_i^{(1)}}{\alpha_i} \right),$$

$$\sum_{j=1}^n \frac{\mu_j |b_{ij}| P_i L_j^g (b_i P_i \tau_i^{(2)} + 1)}{\beta_i b_i} < \mu_{i+n} \left(1 - \frac{b_i P_i^2 \tau_i^{(2)}}{\beta_i} \right),$$

$(i = 1, \dots, n)$.

4. There exist positive numbers $\mu_k, k = 1, \dots, 2n$ *such that*

$$\sum_{i=1}^{n} \frac{\mu_{i+n}|a_{ij}|R_i L_j^f (a_i R_i \tau_i^{(1)} + 1)}{\alpha_i a_i} < \mu_j \left(1 - \frac{b_j P_j^2 \tau_j^{(2)}}{\beta_j}\right),$$

$$\sum_{i=1}^{n} \frac{\mu_j |b_{ij}| P_i L_j^g (b_i P_i \tau_i^{(2)} + 1)}{\beta_i b_i} < \mu_{j+n} \left(1 - \frac{a_j R_j^2 \tau_j^{(1)}}{\alpha_j}\right),$$

$(j = 1, \dots, n)$.

Then the equilibrium (x^*, y^*) *of system (3.1) is globally exponentially stable.*

Proof. By Lemma 2.2 any of the conditions $1 - 4$ implies that C_{BAM} is an M-matrix. □

Remark 3. *Part 3 of Corollary 10 coincides with [15, Theorem 3.1] in the case when* $r_i(t)$ *and* $p_i(t)$ *are constants. In addition to being more general than [15, Theorem 3.1], the result of Theorem 3.3 does not require to find some positive constants, i.e., the check of the signs of principal minors will immediately indicate whether such constants exist or not.*

In the following statement consider system (3.1) without delays in the leakage terms.

Corollary 11. *Suppose* $h_i^{(1)}(t) \equiv t, h_i^{(2)}(t) \equiv t$, *and at least one of the following conditions holds:*

1. $\displaystyle\sum_{j=1}^{n} \frac{|a_{ij}|R_i L_j^f}{\alpha_i a_i} < 1, \quad \sum_{j=1}^{n} \frac{|b_{ij}|P_i L_j^g}{\beta_i b_i} < 1, \quad (i = 1, \dots, n).$

2. $\displaystyle\sum_{i=1}^{n} \frac{|a_{ij}|R_i L_j^f}{\alpha_i a_i} < 1, \quad \sum_{i=1}^{n} \frac{|b_{ij}|P_i L_j^g}{\beta_i b_i} < 1, \quad (j = 1, \dots, n).$

3. There exist positive numbers $\mu_k, k = 1, \dots, 2n$ *such that*

$$\sum_{j=1}^{n} \frac{\mu_{j+n}|a_{ij}|R_i L_j^f}{\alpha_i a_i} < \mu_i, \quad \sum_{j=1}^{n} \frac{\mu_j |b_{ij}| P_i L_j^g}{\beta_i b_i} < \mu_{i+n}, \quad (i = 1, \dots, n).$$

4. There exist positive numbers $\mu_k, k = 1, \dots, 2n$ *such that*

$$\sum_{i=1}^{n} \frac{\mu_{i+n}|a_{ij}|R_i L_j^f}{\alpha_i a_i} < \mu_j, \quad \sum_{i=1}^{n} \frac{\mu_i |b_{ij}| P_i L_j^g}{\beta_i b_i} < \mu_{j+n}, \quad (j = 1, \dots, n).$$

Then the equilibrium (x^*, y^*) *of system (3.1) is globally exponentially stable.*

Consider system (3.1) with $n = 1$:

$$\begin{aligned} \tfrac{dx}{dt} &= r(t)\left(-ax(h_1(t)) + Af(y(l_2(t))) + I\right) \\ \tfrac{dy}{dt} &= p(t)\left(-by(h_2(t)) + Bg(x(l_1(t))) + J\right) \end{aligned} \tag{3.9}$$

where

$$a > 0,\ b > 0,\ 0 < \alpha \le r(t) \le R,\ 0 < \beta \le p(t) \le P,\ |f(u) - f(v)| \le L^f |u - v|,$$

$$|g(u) - g(v)| \le L^g |u - v|,\ 0 \le t - h_i(t) \le \tau_i,\ 0 \le t - l_i(t) \le \sigma_i,\ i = 1, 2.$$

Corollary 12. *Suppose* $\dfrac{aR^2\tau_1}{\alpha} < 1$, *and*

$$\frac{ABRPL^f L^g (aR\tau_1 + 1)(aP\tau_2 + 1)}{\alpha\beta ab} < \left(1 - \frac{aR^2\tau_1}{\alpha}\right)\left(1 - \frac{bP^2\tau_2}{\beta}\right).$$

Then the equilibrium (x^*, y^*) of system (3.9) is globally exponentially stable.

EXAMPLE 2. *Consider the particular case of BAM network described by (3.9)*

$$\begin{aligned}\frac{dx}{dt} &= (20+\mu\sin t)\left[-x\left(t-\frac{1+|\sin t|}{2000}\right)+\frac{1}{720}y\left(t-3\sin^2(t)\right)+10000\right]\\ \frac{dy}{dt} &= (40+\mu\cos t)\left[-x\left(t-\frac{1+|\cos t|}{2000}\right)+\frac{1}{200}y\left(t-2\sin^2(t)\right)+20000\right]\end{aligned} \tag{3.10}$$

for $\mu \geq 0$. Here $L^f = L^g = a = b = 1$, $\alpha = 20-\mu$, $R = 20+\mu$, $\beta = 4-\mu$, $P = 40+\mu$, $\tau_1 = \tau_2 = \frac{1}{1000}$, $A = \frac{1}{720}$, $B = \frac{1}{200}$.

By Corollary 12, system (3.10) is exponentially stable if $\dfrac{(20+\mu)^2}{1000(20-\mu)} < 1$ and

$$\frac{(\frac{20+\mu}{1000}+1)(\frac{40+\mu}{1000}+1)}{720\cdot 200(20-\mu)(40-\mu)} < \left(1-\frac{(20+\mu)^2}{1000(20-\mu)}\right)\left(1-\frac{(40+\mu)^2}{1000(40-\mu)}\right),$$

which is satisfied, for example, if $0 \leq \mu \leq 18$.

Note that for $\mu = 0$ this example coincides with [15, Example 4.1]. It was also mentioned in [15] that exponential stability of (3.10) cannot be obtained using the results of [8, 12, 13, 19], since leakage delays in (3.10) are time-variable. Therefore, our results for the case $\mu > 0$ and with time-varying coefficients and delays are new and applicable to more general models.

4. Discussion and Open Problems. To obtain sufficient stability conditions for nonlinear delay systems four different approaches might be used: construction of Lyapunov functionals, application of fixed point theory, either development of estimations for matrix or operator norms, or making use of some special matrix ($M-$matrix) properties, and the transformations of a given nonlinear system to an operator equation with a Volterra casual operator. While the Lyapunov direct method has been and remains the leading technique, numerous difficulties with the theory and applications to specific systems persist. One of the problems with using the fixed point techniques is the construction of an appropriate map (integral equation) that is sometimes quite difficult or impossible. The technique used in papers [12]–[14] and [20] is based on the construction of Lyapunov functionals.

Perhaps it is worth mentioning that the method applied here to nonlinear system (2.1) is somehow related to the approach used in [9] for linear system (2.10); however, to the best of our knowledge, there are no similar results for (2.1). Remark 1, Example 1, Theorem 3.3 and its corollaries improve and extend results previously obtained for BAM neural networks in [6, 8, 15, 19, 24, 25].

In the framework of this paper, we could not consider all the applications of the M-matrix method to specific models; therefore we outline some particular cases and extensions that might be of interest

for scientists who plan to start future research in this field.

1. Find global stability conditions of system (2.1) for the special cases:

$$F_{ij}(t,x) = \tanh\left(\alpha_i x(g_{ij}(t))\right) \quad \text{and} \quad F_{ij}(t,x) = \frac{1}{1+\alpha_i e^{-x(g_{ij}(t))}}.$$

2. Study global stability for a more general than (2.1) model:

$$\dot{x}_i(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m}\sum_{k=1}^{s} F_{ijk}\left(t, x_j(g_{ijk}(t))\right),\ t\geq 0,\ i=1,\dots,m.$$

3. Derive sufficient stability tests for the equation with a distributed transmission delay
$$\dot{x}_i(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m} \int_{t-\tau_j}^{t} K_{ij}(t,s)F_{ij}\left(s, x_j(g_{ij}(s))\right)\, ds,$$
$t \geq 0,\ i = 1, \dots, m.$
4. Investigate stability of the system with distributed delays in all terms
$$\dot{x}_i(t) = -a_i(t) \int_{t-\sigma_i}^{t} x_i(h_i(s))\, d_sR_i(t,s) + \sum_{j=1}^{m} \int_{t-\tau_j}^{t} F_{ij}\left(s, x_j(g_{ij}(s))\right)\, d_sT_{ij}(t,s),$$
$t \geq 0,\ i = 1, \dots, m.$
5. Obtain sufficient stability conditions for the system with an infinite distributed delay
$$\dot{x}_i(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m} \int_{-\infty}^{t} K_{ij}(t,s)F_{ij}\left(s, x_j(g_{ij}(s))\right)\, ds,$$
$t \geq 0$, $i = 1, \dots, m$, where $|K_{ij}(t,s)| \leq Me^{-\nu(t-s)}$. Generalize this result to the case of exponentially decaying infinite leakage delays as well.
6. Analyze global asymptotic stability conditions of (2.1) when condition(a3) is not satisfied but $\lim_{t\to\infty} h_i(t) = \infty$, $\lim_{t\to\infty} g_{ij}(t) = \infty$, e.g., the pantograph-type delays $h_i(t) = \lambda_i(t)$ for $0 < \lambda_i < 1$. Is it possible to estimate the rate of convergence for some classes of delays?
7. Under which conditions will solutions of BAM system (3.1) with $I_i > 0$, $J_i > 0$ and positive initial functions be permanent (positive, bounded and separated from zero)?
8. Apply the M-matrix method to the following generalization of (2.1)
$$\dot{x}_i(t) = -a_i(t)x_i(h_i(t)) + \sum_{j=1}^{m} F_{ij}\left(t, x_1(g_{ij}^{(1)}(t)), \dots, x_m(g_{ij}^{(m)}(t))\right),$$
$t \geq 0,\ i = 1, \dots, m.$
9. Conjecture:
If C defined by (2.3) has negative off-diagonal entries $a_{ij} \leq 0$, $i \neq j$, and its Moore-Penrose pseudoinverse matrix is nonnegative then system (2.1) is stable.

Remark 4. *To apply the results of the present paper, first use [4, Theorem 9] to reduce exponential stability of equations with distributed delays to exponential stability of equations with concentrated delays. For some other methods see recent papers [1, 2, 3] and references therein.*

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